Can Philosophic Methods without Metaphysical Foundations Contribute to the Teaching of Mathematics?

John Roemischer

Introduction

In the complex teaching paradigm constructed and celebrated in classical Greek philosophy, *geometry* was the gateway to knowledge. Historically, mathematics provided the generational basis of education in Western civilization. Its impact as a disciplining subject was philosophically served by Plato's most influential metaphysical involvement with the dialectical interplay of form and content, ideas and images, and the formal, hierarchic divisions of reality. Mathematics became a key--perhaps *the* key--for the establishment of natural, social and intellectual hierarchies in Plato's work, and mathematical capacities became synonymous with power--the power of abstraction needed to effect and control change (cf. Boisvert 153). It was the provenience of the academic demands, levels, and achievements celebrated as culturally Western. From ancient Athens to medieval Europe, it became the principle of interactional/hierarchic selection: *Similia similibus cognoscuntur*--that is, only those things alike (by nature) can interact cognitively; this launched the notion that mathematical ability is the best test for objectively determining who does and who does not qualify for a social, political, and intellectual meritocracy.

Why mathematics? The capacity to turn interactive experiences with nature into a formal framework, as Euclid did, was inspirational. It generated the architectural metaphor of an empowering foundation of knowledge and work, and foundation came to denote a set of fundamentals that were apodictic and irreducible: axioms come to mind, but, more subtly, the highly variegated Greek term of logos, the Greek foundation stone of thought. The quest for metaphysical foundations simply denoted the promotion of a *fundamentalism* that promised the ultimate closure of inquiry. John Dewey's "quest for certainty" attempted to soften metaphysics, to avoid epistemic closure. He noted that mathematical propositions "appear to be true everywhere and at all times," since their meanings are limited to their place in a closed "formal system" in which "transformation(s) within the system are uniform and dependable within that limited scope." And from the perspective of a pragmatist epistemology, he noted that insofar as "mathematical propositions refer to some existential individual or another, they are not dependably applicable" (Hickman, 1998, p.175).

Notwithstanding Dewey's reconstructionist approach to pedagogical foundations, fundamentalist teaching remained the cornerstone of American schooling. While some metaphysical foundations of mathematics emerged in the 20th century that attempted the further reduction of mathematics to logical foundations, the anti-foundational philosophy of American pragmatism proclaimed that logic was the business of experimental inquiry. Critical reaction to foundational theory soon found its way not only into the work of mathematical theorists, but also those who questioned the use of mathematics as a disciplinary model for intellectual/cognitive development. *Dewey's Democracy and Education*, and progressive educators generally, proposed that the fundamentalism implicated in the use of mathematics as a model for metaphysical foundations of education was the bane of schooling, as many reflective adults would concur; it was anticipated that it would soon be rejected, especially since the moral component implicated in fundamentalism is obedience to rule--or, as poet Shelley proclaimed: "Obedience / Bane of all genius, virtue, freedom, truth / Makes slaves of men."

For internal reasons (given the royal status of geometry), the presence of both irrational numbers and a dependence on sense-based illustrations moved Plato to think that mathematics by itself could not compete with philosophic inquiry in its quest for the discovery of the fundamental elements of reality; however, he did see it as a high-level stepping stone (logos) to education. Plato probably felt, as has Reuben Hersh (1997), that, given mathematicians who think themselves qualified to produce philosophy, “no worse metaphysics than theirs is to be found” (Hersh, 1997, p.199). In his celebration of the appearance of a new humanistic mathematics, Hersh finds laudable 20th century pragmatism’s rejection of the myths that have governed mathematical foundations--”unity, universality, certainty”--(*ibid.* 37). Academically, however, Peirce and Dewey, as well as Socrates, lost the fight against Platonism, and all have become fond memories: experiential, cooperative and Socratic/dialogical inquiry are still mentioned, but Plato’s paradigm, his metaphysical a *priorism* is still celebrated in school-related foundations of mathematics. As Hickman notes, Dewey argued that “Experimental science, as well as everyday experience, is replete with cases in which abstract mathematical propositions are too ‘thin’ to apply to experience in all its robustness. It is important to note that Dewey does not think that mathematical propositions, or any other type of propositions, for that matter, are true or false” (Hickman, *loc. cit.*). At best, as products of experiential inquiries, propositions are either “warranted” or “unwarranted assertions.”

But there are philosophic methods of thought that appear to be independent of metaphysical foundations--methods that shy away from the exclusive concentration on textbook subject-matters. The French mathematician/cognitive neuropsychologist, Stanislas Dehaene, suggests that mathematics instruction turn in this different direction:

> Thus bombarding the juvenile brain with abstract axioms is probably useless. A more reasonable strategy for teaching mathematics would appear to go through a progressive enrichment of children’s intuitions, leaning heavily on their precocious understanding of quantitative manipulations and of counting. . .Eventually, formal axiomatic systems may be introduced. Even then, they should never be imposed on the child, but rather they should always be justified by a demand for greater simplicity and effectiveness. Ideally, each pupil should mentally, in condensed form, retrace the history of mathematics and its motivations. (Dehaene, 1997, p.242)

How odd that more than a century after the reign of Herbartianism in schools, we *should* be reintroduced to a softer version of the old principle that, in teaching, “ontogeny should recapitulate phylogeny.”

This paper selectively explores the applicability of two non-foundational philosophic methods of thought to the teaching of mathematical concepts.

Philosophical Foundations of Mathematics

For those contemporary writers on the philosophy of mathematics who have been critical of the enterprise that has been identified as foundations of mathematics, the gripe has been that the metaphysical bias has been to reductively formulate the enduring, essential components that form the framework of what is actually an evolving and variegated area of knowledge. The ostensible purpose of this framework is to provide a portrait of the real nature of this subject matter, though, paradoxically, its tendency toward reductionism has historically invited not only competing, but also mutually exclusive frameworks. In foundations of mathematics, Platonic formalism has strongly competed against constructivism, intuitionism, et al., while, on the academic side, school teaching has, for generations, settled on Plato’s prioritizing of form over content and the production of reductive definitions. Plato’s purpose in his dialogue, *Meno*, as we shall see below, was to dramatize how the use of formally guided problem-solving algorithms in teaching geometry demonstrates the dependence of mathematics on a specific metaphysical foundation--namely, on the essential a *priority* of subliminal ideal forms. Notwithstanding this so-called embedded awareness, math phobia, as experienced by many children, has been an effect of the abstractionist, cognitive distance between their schooling and their initial and growing involvement with content-laden areas of experience--for example, with games of increasing complexity. As Dehaene has further noted, "Playing snakes and ladders may be all children need to get a head start in arithmetic." This ostensibly more "humanizing" approach to mathematics is elaborated in Dehaene's *The Number Sense*: his section on education begins with an excoriation of Platonic foundations and ends with a casual reminder: "In fact, most children are only too pleased to learn mathematics if only one shows them the playful aspects before the abstract symbolism" (p.143). In several places, Dehaene notes that what is generally lost in the teaching of mathematics is its *meaning*--"our schools are often content with inculcating meaningless and mechanical arithmetical recipes into children" (139); and he demonstrates how the practice of teaching arithmetic through standardized textbooks leads to algorithms that are "not correct" (p.133). "Bugs" are introduced which "Only a refined understanding of the algorithm's design and purpose can help. Yet the very occurrence of such absurd errors suggests that the child's brain registers and executes most calculation algorithms without caring much about their *meaning*" (p.133, italics added). Dehaene's example of a "classical error" pertains to the mysterious notion of "carryover":

> A classical error consists in a leftward shift of all carry-overs that apply to the digit 0. In 307 - 9, some children correctly compute17 - 9 = 8, but then fail to subtract the carryover from 0. Instead, they wrongly simplify the task by carrying over the one into the hundreds column; "therefore", 307 - 9 = 208. Errors of, this kind are so reproducible that Brown and his colleagues [Carnegie-Mellon University study] have described them in computer science terms: Children's subtraction algorithms are riddled with "bugs." (Dehaene, 1997, p.133)

Dehaene then examines three philosophic foundations of mathematics: Platonist, Formalist, and Intuitionist, and finds Platonism most questionable. In effect, all the "myths" that Reuben Hersh (*loc. cit.*) finds in foundational thinking (Platonic especially) seem summed up in Dehaene's version, one that finds him more in agreement with Intuitionist foundations than any other. Ironically, mathematics seems to have an unreasonable effectiveness precisely because "mathematical models rarely agree *exactly* with physical reality." It is the human brain that "translates" physical reality "into mathematics" (*ibid.* 251f). If, in essence, Plato's *a priori* abstract forms are not the product of an evolving induction based on "the regularities of the universe," then teaching them as if they were creates the stresses of textbook-driven mathematics. The irony is that "the *unreasonable* effectiveness of mathematics" seems to be its availability to conflicting foundational positions, and this is possibly the least of those reasons that mathematician, Hilary Putnam, hopes will lead to the deliquescence of [metaphysical] foundations of mathematics.

Efforts to humanize mathematics have inspired our question: "Can philosophic methods *without* metaphysical foundations contribute to the teaching of mathematical concepts?" Here, Gadamer's (1982) recognition of the danger of "dogmatic metaphysics," a danger that consists in the historic development of exclusionist methods of thought that are tied to absolutist metaphysical designs, calls for the propagation of what Husserl called a "free imaginative variation." The search for a system that is reducible to a set of fundamental *(a priori)* elements is thereby resisted. A contemporary pedagogic attempt to rejuvenate the Socratic *protreptic* (open-ended) method (cf. Copeland's "Socratic Circles") might work once Plato's metaphysical design is sidetracked. Given this direction, references to actual mathematical experiences can serve to evaluate questionable, but celebrated, reductionist approaches to foundations. As De Millo, Lipton, and Perlis have noted, the formalist attempt, in Russell's *The Principia Mathematica*, to reduce mathematics to a few and fixed rules of transformation, was "the deathblow for the formalist view." "*If the mathematical process were really one of strict, logical progression, we would still be counting on our fingers*" (De Millo, et al., 1998, p.269, italics added).

These comments mark a crisis now fully evident in competitive philosophical foundations of mathematics, as well as a spillover into the field of teaching mathematics; one such effect was the mid-twentieth century pedagogic preoccupation with set-theory. The crises in the field of teaching appear in the strong curriculum discontinuities generated and supported, historically, by Platonic "form/content" hierarchic dualism. The practical results are often paradoxical. For example, given the metaphysical hierarchizing of Platonic (and also the more naturalized Aristotelian) forms, some contemporary specialized middle schools have institutionalized cognitive discontinuities prematurely in the lives of 11 year-old children. Notwithstanding A. N. Whitehead's warning against premature and persistent methodological specialization, offering a single math and science middle school to a limited population in a large city, but then making math and science the general gateway for all higher education, illustrates one toxic effect of Gadamer's repudiated "dogmatic metaphysics." The use of traditional selective testing processes, supported by dogmatic form/content dualisms, dualisms opposed by progressive educators, has generated a strong distaste for schooling; Dewey had to reconstitute the domain of content as the experiential basis of all learning. So-called "subject matter" had to be tailored to meet the criterion of accessibility, while the Platonic denigration of sensory content was a disservice to the pluralism evident in the variegated learning experiences of children. With John Dewey as his topic, James Garrison (1998) has concisely formulated the progressive rejection of the metaphysics of hierarchic content: "*Educational value is not intrinsic to the subject matter. The value of any given subject matter depends on its contribution to the growth of the learner*. Educators and the public at large must learn that there is no one best method of education. There is no one best way to grow" (Garrison, 1998, p. 69, italics added). Though Dewey departed from Plato by arguing that method is never "outside of the material," he nevertheless pointed out that there is a logical sense of method that distinguishes the structure of the subject matter as it appears to the expert or specialist (*ibid.*). The objective of progressive pragmatism was to bypass the finality of the traditional truth criterion by rejecting the notion of non-reconstructible fundamentals. As Raymond Boisvert notes, for pragmatists, "There is no foundational data that simply offers itself up to the inquirer as absolutely fundamental." At best, even those "logical" structures that come with expertise are largely "maps" that must serve some purpose (Boisvert p. 150). Hence, in the history of mathematics, progress has required a *reconstruction* of fundamentals. In time, the 10- finger model has begun to yield to a binary base.

Hilary Putnam's viewpoint in "Models and Reality," originally presented in 1977, is preceded by a more radical statement presented ten years earlier--his essay: "Mathematics without Foundations." In the 1930's (especially, and perhaps ironically, in the work of Kurt Gödel), Platonism became a dominant force in foundations of mathematics, one that Paul Bernays succinctly identified as abstractionism: Platonism (especially in mathematics) views the object "as cut off from all links with the reflecting subject" (Bernays, 1983, p. 258). Putnam (1967) does "not think mathematics is unclear; [he doesn't] believe mathematics has or needs 'foundations'. The much touted problems in the philosophy of mathematics seem to me, without exception, to be problems internal to the thought of various system builders. . .[T]he various systems of mathematical philosophy, without exception, need not be taken seriously" (Putnam p. 295). Keep Putnam in mind while noting Collingwood's view that Plato failed "to drive deep enough the distinction established by himself between philosophy and mathematics." Perhaps Plato could not comply with Collingwood's desire, for without mathematics, Socrates could not have provided the evidence Plato sought for his major premise, vis-à-vis, the congruence of *a priori* knowledge and mathematics.

Philosophic Method: Disciplinary Process or Subject-Matter?

The question under review here is whether philosophic methods, as methods of "theoretical thought," can have a productive impact on the practice of teaching mathematical concepts. It should be noted that the term "philosophic method" has a controversial history, one briefly summarized in Gadamer's (1982) *Truth and Method*. By placing philosophy in the hands of two of its famous teachers, Kant and Hegel, Gadamer noted that method emerges when "dogmatic metaphysics" gives way to philosophy as a process of conceptual clarification--in effect, a disciplinary process of thought rather than a subject-matter. Of the highest historical importance is Gadamer's claim that "prekantian dogmatic metaphysics" is still characteristic of "*the modern ages of non-philosophy*" (Gadamer, 1982, p. 424, italics added).

Gadamer's critical reaction to dogmatic metaphysics has to include approaches to methodology, since method is a term with built-in fallibilities--it is never "free of all prejudices." Even "the certainty imparted by the use of scientific methods does not suffice to guarantee truth." Since, "in the knowing involved in [the human sciences], the knower's own being is involved, [this certainly marks] the limitation of 'method', but not that of science." However, Gadamer does not abandon truth: "what the tool of method does not achieve must--and effectively can--be achieved by a *discipline of questioning and research, a discipline that guarantees truth*" (ibid. p. 446, italics added).

In J. M. Bochenski's (1965) *The Methods of Contemporary Thought*, philosophic methods are designed precisely as "disciplines of questioning and research." Though Bochenski does not specify it, his distinction between "practical thought" and "theoretical thought" might best be understood in terms of a distinction in the logic of types of questions, as we shall see below, as well as the important distinction between "algorithmic" and "dialectical mathematics," also forthcoming.

Bochenski attempted to direct attention to those methods that were not generated from metaphysical foundational philosophies, methods that might serve theoretical thought without the paradox generated in the crises of foundational systems. To the extent that teaching practice involves more than management and strategy, and since it could be a process for promoting theoretical thought, the paradox is that those philosophic foundational conflicts that supposedly govern teaching from above or outside the practice cannot therefore be resolved in and through the foundation-governed practices themselves. This defines the general crisis in foundations, a condition concerning which Socrates' ironic claim of "ignorance" was a forewarning. The philosophic point is that teaching could be the arena in which different non-foundational philosophic methods can all be critically employed as disciplinary methods--that is, in Gadamer's sense of a "discipline of questioning and research." If these philosophic methods can all work independently of foundational metaphysics, and interdependently as methods of teaching, then, arguably, the so-called crisis in foundations of mathematics can be avoided, at least from a pedagogical perspective. These disciplines, then, become the application of philosophical methodology to the teaching of mathematical concepts.

Examples of traditional dogmatic metaphysical approaches, some that move deductively from metaphysical positions to teaching theory, can be found in Nelson Henry's (editor) (1955) *Modern Philosophies and Education*. Here, the content is radically different from the non-foundational methods that Bochenski identifies. Bochenski's "contemporary methods of thought" do not bear the labels that identify those metaphysical systems identified by Henry: Realism, Thomism, Christian Idealism, Marxism, et al. In mid-20th century, as American schools faced school challenges from abroad, foundational metaphysics was hidden under a cloak consisting of a mechanized psychology of learning that was presumed given, and a homogenized, pseudo-scientific lesson planning process utilizing behavioral objectives. Bochenski's "methods of contemporary thought," which are not theories about teaching, but are philosophic methods that can only do service when practiced as diverse methods in teaching, failed to penetrate the politicized teacher-training colleges. Notwithstanding the efforts of pragmatists to neutralize dualist distinctions--theory vs. practice, form vs. content, truth vs. meaning, among others, metaphysical dualisms have been enshrined in American education. While schools and teachers have seesawed between so-called methods of practice, philosophic foundations of education have futilely moved from dogma to dogma. Students tend to see no relationships, no language continuities, in the thought processes used in their atomized curriculum; they tend to confuse teaching with presentation, re-presentation, and repetition--in effect, students tend to stagnate in a mimetic learning modality rather than move to conceptual thinking. The challenge to Bochenski's distinction between "methods of practical thought" and "theoretical thought" consists of a rather odd observation, vis-à-vis, that his "methods of contemporary thought" (four in all) are, from a pedagogical point-of-view, both practices and theories. Here, philosophic methods of thought are teaching processes.

Non-Foundational Teaching: A Discipline of Questioning and Research

Once an area of research lends itself to the closure provided by a metaphysical foundation, it tends to restrict the most relied-on instrument available to both philosophy and teaching, vis-à-vis, questioning as a process. And when the area of research becomes a subject-matter, the use of questioning becomes less a process of conceptual exploration and more a device for reinforcing learning of what is given. Work in the logic of questions offers this distinction: “whether questions” (pertaining to a yes/no, right/wrong approach to facts and definitions in subject matters) are different from “which questions” (pertaining to conceptual exploration of meanings). Most teachers, governed by the restrictive framework of a metaphysical foundation (most likely Platonism), will tend to barrage children with rows of yes/no or true/false “whether questions.” The pedagogic instrument called questioning simply serves to ascertain the extent to which the child can conform to a closed system of identifications and mechanized solutions to problems; more often than not, routine exercises are honorifically called problems.

Space permits just two examples of Bochenski’s philosophic methods: they are precisely “disciplines of questioning and research.” If we take Bochenski’s discussion of the philosophic Semiotic Methods, and then proceed to John Wilson’s (1963) section on “General Justification of Linguistic Analysis,” we can find a comparable “justification” in his *Thinking with Concepts*. This pedagogic/philosophic work actually introduces a “method of practical thought”--one that puts theoretical thought to work in the advancement of concepts.

Wilson begins with a distinction that clarifies Bochenski’s--the distinction of teaching thinking as a disciplinary process rather than a subject-matter. His philosophic method, the analysis of concepts, as a practical method of semantic discovery, can be distinguished from such typical school methodized subjects as Latin prose, varieties of mathematics, German, et al. Wilson notes, “Often we can look up the right answers to questions in these subjects, by referring to a dictionary, or a grammar, or an authoritative textbook. But none of this applies to the techniques outlined [in this analysis of concepts].” Wilson’s questions of concepts are not “whether-or-not” questions of such text-based facts; they are “which” questions, questions that open a variety of pathways to conceptualization. Hence, it might help to preface Wilson’s work with a brief reference to the logic of questions.

Belnap and Steel (1976), in their *The Logic of Questions and Answers*, state that there are questions, for example, “What is a number?”, that “give little indication as of what would count as an answer” (Belnap & Steel, 1976, p.12). It might appear, then, from the Belnap and Steel discussion, that conceptual questions and the classroom infatuation with definitional questions are different in kind. Is the instructional pursuit of definitions, so commonplace in teaching mathematics, more caught up with answers than with an understanding of the questions themselves?

“Elementary questions can be classified into two sorts, depending on how many alternatives they present”: “whether” questions allow for a few or finite number of alternative answers, and these are often “explicitly listed in the question”--e.g., “Does brass contain more tin than copper?” On the other hand, “which” questions contain a potentially large number of cases--and these are not in the question but are presented as part of some condition or matrix--e.g., questions in ethics as to which principles are ethical and why, and questions in mathematics as to which symbol is a number and why. The point is that while “which” questions also apply to mathematical reasoning (e.g., “which numbers are imaginary?”), in Plato’s *aporetic* (inconclusive) dialogues, choices were inconclusive precisely because limited from the standpoint of the logic of questions: as Colin McLarty (2005) notes, Socrates was essentially concerned with the destruction of hypotheses; the ostensive direction was not to dictate secure definitions, but to filter unexamined or sensory-based opinions. Plato’s denigration of mathematics, when viewed from the standpoint of philosophic methods of thought, involved its inability to dispense with sensible diagrams. This, unhappily, gave “whether” questions a dominant place in mathematical inquiry. In essence, notwithstanding Plato’s attempt to give mathematics a central place in his educational curriculum, he felt that mathematics was “irremediably defective” (Gonzalez, 1998, p. 377). Nevertheless, and ironically, Plato’s metaphysical Realism led Socrates’ dialectical struggle to get “which” questions fully developed through dialogical/dialectical processing to resort to more restricted “whether” questions. The fault was not in his determination. “Ordinary yes-no questions are whether-questions, for from the question we can easily directly recover the statements presented as alternatives. . . .Any finite set of formulas is called an abstract whether-subject" (Belnap & Steel, 1976, 19f, italics added). In Plato's dialogue, *Meno*, Socrates' sensory approach to teaching geometry by a presentation of two similar, yet different, geometric figures--two squares, one superimposed on the diagonal of the other--resulted in a geometric problem in which the two objects appeared to be related and yet unrelated (different) at one and the same time. The solution to the problem involved the formulation of a hypothesis to determine whether these objects were related or unrelated in such a way as to account for their appearance. In effect, the geometric problem that had to be solved was related to "the set of *alternatives* it present[ed]" which "is defined as identical with its subject." Plato's denigration of mathematics, Gonzalez opines, was that its involvement with sensible diagrams raised the suspicion that it could never "cease to be 'hypothetical'" (*loc. cit.*). The puzzling issue is why Socrates' questioning process in his mathematics lesson in *Meno* was not different in design when he turned to his dialogical/dialectical inquiries into meaning generally; that is, why did he settle for mathematically-styled definitions rather than concepts?

But here lies buried the final Socratic irony. As Collingwood noted: notwithstanding the fact that Plato and Socrates at least *attempted* to distinguish philosophy and mathematics, Socrates used mathematics as a "*model for dialectical reasoning*." It moved Socrates, in his dialogical quest for the meaning of concepts, in the direction of definitional knowledge that could be offered as hypotheses. "Whether" questions prevailed, that is, as Collingwood notes, when Socrates "asked himself or his pupils to *define* a concept, the model which he held up for imitation was *definition as it exists in mathematics.* This no doubt accounts for his [repeated philosophic] failure; and it also accounts for the tendency which exists at the present time to deny that philosophical concepts admit of definition" (Collingwood, 1933, 92ff, italics added). However, while lacking the tools of modern philosophic methods and proclaiming ignorance, at least Socrates' use of mathematics as a model for dialectical reasoning allowed his students the opportunity to contribute to the dialogue. Hence, philosophy's quest for concepts seemed to profit from the mathematical production of definitional possibilities, without which Plato's dialogues would have faltered. What could be gained from a classroom discussion of philosophical method and mathematical thinking is a comparison and contrast of the logic of concepts and the logic of definitions, and a comparison and contrast of forms of questioning.

In "a mathematical concept," Collingwood notes, "some one attribute is essential and the others flow from it" (*ibid.*, 99). This conceptual quest for the essences of the meanings of things has haunted Western philosophy--a quest that phenomenology had to re-open by its methodological insistence on free imaginative variations. Hence, notwithstanding Plato's reservations, mathematical concepts might now be more open to exploration than he thought. While Plato, in time, made an effort to account for complexity by distinguishing between things in terms of a scale of forms, that effort was better developed and elaborated by Aristotle. As Collingwood notes, instead of examining the range of a concept, Plato's Socrates kept asking for "a unitary definition" of a term like "virtue" (cf. Socrates' encounter with Glaucon in Plato's Dialogue, *Republic*); for Aristotle, such concepts ranged from lowest to highest manifestations--for example the form of "virtue" found in a slave to higher and higher forms (*Ibid.* 10).

Free Imaginative Variations in Didactical Philosophy

Evidence to the effect that Platonism is still in the ring fighting for dominance in philosophical foundations of mathematics is stressed in Stanislas Dehaene's aforementioned study. By grounding mathematics in history, Dehaene does precisely what Mary Warnock (1994) prescribed in her *Imagination and Time*. He states that "The history of number notations is hard to reconcile with the Platonist conception of numbers as ideal concepts that transcend humankind and give us access to mathematical truths independent of the human mind." And, contrary to the "Platonist mathematician Alain Connes," Dehaene argues that mathematical objects are not "untainted by cultural associations." If an "abstract concept of number" had been the driving force of the "evolution of numeration systems. . .as generations of mathematicians have noted," then, Dehaene points out, "binary notation would have been a much more rational choice than our good old base 10. . .[a base] due to the contingent fact that we have ten fingers." And citing Karl Popper, Dehaene writes, "The natural numbers are the work of men, the product of human language and of human thought" (Dehaene, 1997, p. 117).

A .E. Taylor's (1936) masterful work on Plato's philosophy provides a condensed version of Socrates' geometry lesson to Meno's slave-boy: "The point insisted on is that the lad starts with a false proposition, is led to replace it by one less erroneous, and finally by one which, so far as it goes, is true" (Taylor, 1936, p. 137). Socrates' series of bi-value "whether" questions led the slave-boy to a correct conclusion; however, it was a demonstration and not the type of dialogical/dialectical interaction that required questions conducive to more open-ended conceptual inquiries.

From a radical empiricist's standpoint, D. W. Hamlyn (1978) rejects Plato's concept that all learning is recollection--that is, is built on *a priori* foundations. It might be argued that *a priori* truths can be found in geometry, however not in cases "where what has to be learnt is an empirical truth." All that the slave-boy has to do "is to work out the logical consequences of what he already knows, even if he has to be jogged along in the process. The 'demonstration' is therefore something of a fraud and it cannot be taken as showing that all learning is recollection" (Hamlyn, 1978, p. 6). However, since Hamlyn accepts Aristotle's notion that "Socrates was the first to use induction, . . .to have *used* instances or examples to give point to a general principle rather than abstracting a principle in the instances," the implication is that Socrates was much more at home in the domain of empirical reality than was Plato. Plato's theory of *a priori* knowledge might find comfort in the domain of geometry, and this might account for its popularity in Platonic mathematical foundations, but it might also account for the essential irrelevance of such foundational thinking to Socrates' inductive approach to teaching mathematics in *Meno*. Thus, following K. R. Popper's work, Perkinson notes that "The notion of making the slave boy aware of his ignorance I take to be socratic; but the notion of recollection of ideas (the theory of innate ideas), I take to be platonic" (Perkinson 10n).

But the more noteworthy method of Socratic thought was not a process of direct induction from examples; rather, it was his interest in dialectical thought, a process that has surfaced as broad-based dialectical mathematics, and this would appear to come close to a concept-driven humanistic approach to mathematics. This reflects Socrates' original, more open *protreptic* (extended, "turning-toward"...) procedure: the distinction between dialectical mathematics and algorithmic mathematics seems as close an example of how one might distinguish Socrates' dialectical method of thought from his mathematics lesson in *Meno*. From Peter Henrici's work in applied mathematics comes the following elaboration of this distinction: "*Algorithmic mathematics* is a tool for solving problems. Here we are concerned not only with the existence of a mathematical object, but also with the credentials of its existence. . . *Dialectic* mathematics *invites contemplation. Algorithmic mathematics invites action. Dialectic mathematics generates insight, Algorithmic mathematics generates results*" (cited in Davis & Hersh, 1998, p. 183, italics added). (Space prohibits an elaborate review of examples provided by these authors).

Returning to Bochenski's discussion of Semiotic methods, Wilson's *Thinking with Concepts* explores this dialectical method in the philosophical analysis of concepts: "which" questions are embedded in the steps students can use to explore and clarify meanings--an exploration that might take them across a variety of subject fields. A brief and somewhat limited example of the procedure, imaginatively applied to mathematical concept development, is as follows:

Conceptual questions and language analysis:

a) *Model cases:* Provide a clear-cut case of a mathematical 'number'.. . (a set of cumulative symbols with predecessors and successors).

b) *Contrary cases*: Provide a negative case.. . .(non-cumulative symbols with no predecessors or successors . . .)

c) Related cases: Provide a comparable case. .(a symbol with no natural predecessor and merely finite conventional successors--

a,b,c...)

d) Borderline cases: Provide an analogous case. . .(a finite set of successive but non-cumulative symbols--a musical scale.)

e) Invented cases: Provide an invented case . . . ? (A set of non-cumulative symbols that can be differentiated and yet overlapped: for example, enharmonic musical tonal notations: *b* and *c flat*...)

Another example: "Provide a clear case of a number that can be expressed as a comparative distinction." (2--dual). A related case: (3 or more--plural). Invented case: (?). Question: why not past 2 or 3? (Dehaene, 1997, p. 93.).

Turning to the Phenomenological Method, some of the most productive work on teaching mathematical concepts phenomenologically can be found in the Oxford journal, *Philosophia Mathematica* (cf. Mary Leng, vol. 10, 2002: "Phenomenology and Mathematical Practices"). The purpose of Phenomenology, as Edmund Husserl defined it, was to free the imagination--through teaching, we might add here. In Richard Schmitt's (1967) outline, Husserl's philosophic method is formulated as follows:

> Husserl talked about a procedure that he called "free imaginative variation," comparable to what Anglo-American philosophers call the method of "counter-examples" [cf. Wilson above). Here we describe an example and then transform the description by adding or deleting one of the predicates contained in the description. With each addition or deletion we ask whether the amended description can still be said to describe an example of the same kind of object as that which the example originally described was said to exemplify. Sometimes we shall have to say that if we add this predicate to the description or take that one away, what is then described is an example of a different kind of object from that exemplified by the original example. At other times the additions or deletions will not affect the essential features of the kind of object exemplified by the different examples. (Schmitt, 1967, p. 141)

In order to free the imagination, phenomenological method moves in three steps: "(1) [the recognition that] phenomena are essences. (2) phenomena are intuited, (3) phenomena are revealed by 'bracketing' their existences" (*ibid.*). As a bare outline, this already suggests a remedy for the narrow, inductively abbreviated, answer-driven approach to turning concepts into definitions. A great variety of directions are open to inquiry, since the method does not merely settle on a single definition as the product of an inductive abstraction. What is the essence of number that allows it to expand beyond the limits of ordinary language? One soon discovers that the restrictive borders of those subject matters that constitute traditional curriculum do not govern the work of phenomenological method. Some of the most productive work for phenomenological teaching has come under the title, "didactical phenomenology," and here perhaps one of the most commendable and possibly revolutionary works is Hans Freudenthal's (1983) *Didactical Phenomenology of Mathematical Structures*.

In his first chapter, Freudenthal approaches mathematics through the phenomenology of extension by moving directly into the concept of length. In ten pages, he shows how the concept evolves, first through phenomenological analysis of length's variety of manifestations, and then, through mathematical symbolization and didactic phenomenology, into an object to be taught. His range is wide: from his treatment of the concept length, he moves to sets, natural numbers, fractions. . ., and finally to algebraic language and functions. Freudenthal's critique of contemporary mathematics instruction is that it lacks a phenomenological base, and is therefore dependent on psychology:

> All the psychological investigations. . .which I know about suffer from one fundamental deficiency: investigations on mathematical acquisitions (at certain ages) have involved the related mathematical structures in a naive way--that is they lack any preceding phenomenological analysis--and as a consequence, are full of superficial and even wrong interpretations.

> The lack of a preceding didactical phenomenology, on the other hand, is the reason why such investigations are designed in almost all cases as isolated snapshots rather than as stages in a developmental process. (Freudenthal, 1983, p. 10)

Conclusion: The Socratic/Platonic Chasm

In his *An Essay on Philosophical Method*, Collingwood notes: "when Aristotle asked himself what contribution Socrates had made to philosophy, he answered in terms implying that, in his opinion, Socrates was essentially the *inventor of method*--not, we might add, the inventor of metaphysics (Collingwood, 1933, p. 10, italics added). If Platonic metaphysical foundations traditionally supported mind-based, social-class hierarchic divisions, then one might reasonably argue that Platonic foundations was not what Socratic teaching method had in mind. Socrates' purpose was an unlocking teaching method--a *protreptic* (indefinitely extended) philosophic method using an *elenctic*, that is, a destructive/constructive dialogic/dialectic question-and-answer process to explore meanings and clarify concepts. The point was not to teach truths that were meaningless, but to realize that if a concept isn't meaningful, its truth is immaterial. In Socrates' attempt to teach mathematics (*Meno* Dialogue), Socrates seemed to move away from the destructive/analytic side of the *elenchus*, but as Gonzalez argues, Plato, from a methodological standpoint, never abandoned this fundamentally oppositional response to the constructivism embedded in mathematics (Gonzalez, 1998, p. 330).

In a contemporary attempt to bring Socratic philosophy into the lives of young people, Matthew Lipman attempted to demonstrate that the problem of closure in contemporary schooling was, ironically, largely due to a loss of interest in the contribution that philosophy made to methods of thought. His discussion, in "The Role of Philosophy in Education for Thinking" (1988), supports Collingwood's critical analysis of the abstractionism and closure embedded in typical schooling. Lipman identifies this drive toward closure as "the rationalistic disposition of the non-philosophic disciplines..." He points to the effects of what happens when academic disciplines actually shed their commitments to Socratic teaching--that is, ironically, shed the authentic disciplinary process from their disciplines. From a Socratic standpoint, what's left is a static, closed system of abstractions--literally, a subject matter merely designed for study: in effect, the classroom becomes a non-Socratic, Platonistic portrait of the world. And here, too, is Mary Warnock's prophecy that teaching--perhaps especially teaching mathematics--could come alive through history. "For a discipline to stay alive, it must re-animate the thinking that went into it at its inception and subsequent formation" (Lipman, 1988, p. 33).

If we generalize Lipman's claim and apply it to contemporary fields of foundations of mathematics (Formalism, Logicism, et al.), it might account for the fact that contemporary foundations--except for one, Platonism--have had little if any impact on classroom mathematics. It should come as no surprise that Platonism has been the dominant, but troublesome, foundational voice as well as the non-problematic basis of those academic disciplines Lipman derides. Despite the views that teachers convey to their students, Collingwood would agree that, by definition, a discipline can never be ultimately non-problematic. The foundational notion of ultimate closure produces the crises in foundations. While philosophy has served as a source of foundations of mathematics, philosophy cannot serve as a foundation of itself. "In a philosophical inquiry what we are trying to do is not to discover something of which until now we have been ignorant, but to know it better in the sense of coming to know it in a different and better way" (Collingwood, 1933, p. 11). Arguably, a corollary proposition should be that no prior philosophic foundation is required for Socratic teaching to occur, since needing one would contradict the purpose of Socratic inquiry. Furthermore, if the subject-matter being taught is still evolving, as most are, then no philosophic foundation should be applied to it that creates closure. In effect, Socrates ironically proclaimed his "ignorance" of "models and reality," since he sensed that philosophy was a method of inquiry to be mastered through practice rather than a subject matter merely to be learned. This Socratic insight might be the most productive way to connect philosophic method to mathematics education; though in itself limited, it has opened the door to the possible use of Socratic philosophy in all teaching domains. And here, also, is the reason that underlies our earlier suggestion, vis-à-vis, that philosophic methods that might support the teaching of mathematics should be sought in non-foundational/non-metaphysical philosophic methods of thought.

Contemporary teaching, Peter Senge (2000) points out, is governed by a foundational fiction: the use of a fictional device to terminate imaginative thought. Schools create the world of "as if..." Schools "teach *as if* they are communicating truth. Kids learn 'what happened' in history, not an accepted story about what happened. Kids learn scientific truths, not models of reality that have proven useful. They learn the one right way to solve a particular problem, not the complexities of different perspectives. As a consequence, students' tolerance for ambiguity and conflict is diminished, and their critical thinking skills fail to develop" (Senge, 2000, p. 46).

This helps us outline the components of Socrates' attempt to discover meaning through a foundations-free teaching method. In each instance, what is sidelined is the typical quest for systemic closure. First, a defining characteristic of Socratic *protreptic* method is its attention to on-going, open criticism--or critique. Second, as Perkinson notes, criticism can be further defined as an acknowledgement of that most fearsome epistemic challenge, vis-à-vis, the possibility of error. (Perkinson, p. 5, et passim.) Third, given the strong possibility of error, a constant attention to the avoidance of authoritarianism is called for. Fourth, the authoritarian approach to teaching method consists of the troublesome premise that teachers both know and are communicating truth. Paradoxically, avoiding infallibility is the primary mark of a good teacher.

Whatever his failings, Socrates did bring methodological insight into the world of education. For good philosophical reason, it was John Dewey who celebrated his effort. From the standpoint of the humanization of the mathematics classroom, John Dewey went beyond Collingwood to distinguish Socrates' pedagogical contribution from Plato's metaphysical approach to the problem that Collingwood addresses. In Hickman's (1998) attempt to place Dewey's concept of abstraction into the context of his theory of experimental inquiry, Dewey was at least cognizant of Socrates' noble effort: "Socrates' attempt to get his fellow Athenians to engage in hypothetical reasoning constituted a great step forward in the history of inquiry. *But Plato made the opposite mistake: when he began to treat abstractions as metaphysical entities, he set an unfortunate course for twenty-five hundred years of Western philosophy*" (Hickman, 1998, p. 174, italics added).

In Dewey's own philosophic critiques of metaphysical foundations--his critiques of abstractionism, of the a priori, of value and truth antecedence, of discontinuity--he attempted to restore the experiential/historic interconnections of mathematics; these critiques might now serve mathematics education from the standpoint of methods of inquiry. Bringing Socratic dialogical/dialectical and 20th century non-foundational philosophic methods in to support humanistic mathematical teaching processes might finally put an end to school-related mathephobia.

This paper concludes, in the spirit of its beginning, with Stanislas Dehaene's prescient attempt to bring the Socratic search for meaning back to life: "The flame of mathematical intuition is only flickering in the child's mind; it needs to be fortified and sustained before it can illuminate all arithmetic activities. *But our schools are often content with inculcating meaningless and mechanical arithmetical recipes into children*" (Dehaene, 1997, p. 139, italics added).

References

Belnap, N. D., Jr. & Steel, T.B. Jr. (1976). *The logic of questions and answers*. New Haven: Yale University Press.

Bernays, P. (1983). On platonism in mathematics. In P. Benacerraf and H. Putnam (Eds.), *Philosophy of math ematics, selected readings*. Cambridge: Cambridge University Press.

Bochenski, J. M. (1965). *The methods of contemporary thought*. Dordrecht, Holland: D. Reidel Publishing Co.

Collingwood. R.G. (1933). *An essay on philosophical method*. Oxford: At the Clarendon Press.

Davis, P.J. & Hersh, R. (1998). *The mathematical experience*. Boston: Houghton Mifflin Co.

Dehaene, S. (1997). *The number sense: How the mind creates numbers*. Oxford: Oxford University Press.

De Millo, R.A., Lipton, R. J., & Perlis, A.J. (1998). Social processes and proofs of theorems and pro grams. In Thomas Tymoczko (Ed.). *New directions in the philosophy of mathematics*. Princeton: Princeton University Press.

Freudenthal, H. (1983). *Didactical phenomenology and mathematical structures*. Dordrecht, Holland: D. Reidel Publishing Co.

Gadamer, H. (1982). *Truth and method*. New York: The Crossroad Publishing Co.

Garrison, J.W. (1998). John Dewey's philosophy as education. In Larry A. Hickman, *Reading Dewey: Interpreta tions for a postmodern generation.* Bloomington, Indiana: Indiana Univeristy Press.

Gonzalez, F.J. (1998). *Dialectic and dialogue: Plato's practice of philosophical inquiry*. Evanston, Illinois: Northwest ern University Press.

Hamlyn, D.W. (1978). *Experience and the growth of understanding*. London: Routledge & Kegan Paul.

Henry, N.B. (Ed.). (1955). *Modern philosophies and education* (The Fifty-fourth Yearbook of the National Society for the Study of Education). Chicago: The University of Chicago Press.

Hersh, R. (1997). *What is mathematics, really?* Oxford: Oxford University Press.

Hickman, L. (1998). Dewey's theory of inquiry. In Larry Hickman (Ed.) *Reading Dewey: Interpretations for a post modern generation.* Bloomington: Indiana University Press.

Lipman, M. (1988). *Philosophy goes to school.* Philadelphia: Temple University Press.

McLarty, C. (2005). 'Mathematical platonism' versus gathering the dead: What Socrates teaches Glaucon." *Philosophia Mathematica* 13(2): 115-134.

Perkinson, H.J. (1971). *The possibilities of error: An approach to education*. New York: David McKay, Inc.

Perkinson, H.J. (1980). *Since Socrates: Studies in the history of educational thought*. New York and London: Long man.

Putnam, H. (1983a). Mathematics without foundations. In P. Benacerraf and H. Putnam (Eds.). *Philosophy of mathematics, selected readings.* Cambridge: Cambridge University Press.

Putnam, H. (1983b). Models and reality. *Philosophy of mathematics, selected readings*. Cambridge: Cambridge University Press.

Schmitt, R. (1967). Phenomenology. In Paul Edwards (Ed.), *The encyclopedia of philosophy*, Vol. 6 (pp. 135-151). New York: Macmillan Publishing Co., Inc. & The Free Press.

Senge, P. (2000). *Schools that learn: A fifth discipline resource.* New York: Doubleday.

Taylor, A.E. (1936). *Plato: The man and his work.* New York: The Dial Press Inc.

Warnock, M. (1994). *Imagination and time*. Oxford: Blackwell Publishers Ltd.

Wilson, J. (1963). *Thinking with concepts*. Cambridge: At the University Press.

Address Corresspondences To:
John Roemischer, Emeritus
SUNY Plattsburgh
johnroemischer@gmail.com